Assignment 2.

This homework is due *Thursday*, September 17.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much.

1. Quick reminder

A relation $R \subseteq X \times X$ is an *equivalence* relation if it is

- reflexive: $\forall x \in X\ xRx$,
- symmetric: $\forall x, x' \in X$ if xRx' then $x'Rx$,
- transitive: $\forall x, x', x'' \in X$ if xRx' and $x'Rx''$ then xRx''.

A relation $R \subseteq X \times X$ is a *partial ordering* (partial order) if it is

- reflexive: $\forall x \in X\ xRx$,
- antisymmetric: $\forall x, x' \in X$ if xRx' and $x'Rx$ then $x = x'$,
- transitive: $\forall x, x', x'' \in X$ if xRx' and $x'Rx''$ then xRx''.

A partial order is called *total* if $\forall x, x' \in X\ xRx'$ or $x'Rx$.

2. Exercises

(1) Let E be the set of all infinite sequences of 0 and 1. Prove that E is equipotent to $E \times E$.
Hint. Consider the map $(a_1, a_2, a_3, \ldots) \mapsto ((a_1, a_3, a_5, \ldots), (a_2, a_4, a_6, \ldots))$.
Comment 1. Actually, the same is true for any infinite set.
Comment 2. With a little technical work, one can show that $[0, 1]$ is equipotent to E. Thus, this problem shows that the segment $[0, 1]$, contrary to intuition, is of the same cardinality as the square $[0, 1] \times [0, 1]$.

(2) Determine whether the following are equivalence relations on X:
  (a) $X = \mathbb{R}$, $x \approx y$ if and only if $|x - y| < 0.1$.
  (b) $X = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$, $(n_1, m_1) \equiv (n_2, m_2)$ if and only if $n_1 m_2 = m_1 n_2$. (What is a good name for this? There are at least two answers.)
  (c) $X = \{f \mid f : A \to \mathbb{R}, A \subseteq \mathbb{R}\}$ (the set of functions from a subset of $\mathbb{R}$ to $\mathbb{R}$),
  functions $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ "partially agree" if and only if their restrictions on $A \cap B$ are equal:
$$f \sim g \quad \text{iff} \quad f|_{A \cap B} = g|_{A \cap B}.$$

(3) Determine whether the following are partial orders on X:
  (a) $X = \mathbb{R}_{>0}$ (positive reals), $x \ll y$ if and only if $y/x > 10$.
  (b) $X = \{[a, b] \mid a, b \in \mathbb{R}, a \leq b\}$ (closed intervals), $[a, b] \preceq [c, d]$ if and only if $a \leq c$ and $b \leq d$.
  (c) $X = \{f \mid f : A \to \mathbb{R}, A \subseteq \mathbb{R}\}$. For functions $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$, put $f \preceq g$ if and only if
  $A \subseteq B$ and $f = g|_A$; in other words, if and only if f is a restriction of g.

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(4) (1.4.27) Is the set of rational numbers open? Is the set of irrational numbers open?

(5) Prove the classification of open sets in $\mathbb{R}$, as outlined below.
Theorem. Every open set in $\mathbb{R}$ is a countable union of disjoint open intervals.

Proof. (a) Consider the following relation on a given open set $\mathcal{O}$: $x \sim y$ if and only if $[x, y] \subseteq \mathcal{O}$ or $[y, x] \subseteq \mathcal{O}$ (depending on whether $x \leq y$ or $y \leq x$). Show that it is an equivalence relation.

(b) Denote the equivalence class of $x \in \mathcal{O}$ by I_x. Consider the extended reals $a_x = \inf I_x$, $b_x = \sup I_x$. Show that $a_x, b_x \notin I_x$. (*Hint:* Use $\mathcal{O}$ being open.) Conclude $I_x = (a_x, b_x)$, i.e. I_x is an open interval.

(c) Conclude that $\mathcal{O}$ is disjoint union of open intervals I_x. (*Hint:* I_x are equivalence classes.)

(d) Show that that the union must be countable. (*Hint:* Every I_x must contain an (its own) rational point.)

□

(6) (1.4.37) Show that every open set in $\mathbb{R}$ can be represented as a countable union of closed sets. (*Hint:* Use classification theorem.)

3. Extra exercises

Problems below will only go to the numerator of your grade for this homework. Also, the due date on these problems is the last day of classes, December 11. That is, you can submit any of these problems any time before classes end.

(7) Give an example of a family $\mathcal{F}$ of distinct subsets of a countable set s.t. the following two conditions hold:
- $\mathcal{F}$ is uncountable.
- $\mathcal{F}$ is a *chain* (totally ordered set) with respect to set inclusion, i.e. for every two subsets A, B in the family $\mathcal{F}$, either $A \subseteq B$ or $B \subseteq A$.

(8) Give an example of a family $\mathcal{F}$ of distinct subsets of a countable set s.t. the following two conditions hold:
- $\mathcal{F}$ is uncountable.
- For any $A, B \in \mathcal{F}$, the intersection $A \cap B$ is *finite*.

(9) Suppose X and Y are two sets. Prove that if there is an injection $f : X \to Y$, and an injection $g : Y \to X$, then there is a bijection $\varphi : X \to Y$.
Comment. This problem essentially asks to prove that "injects into" is a partial order on classes of equipotence, so that it makes sense not only to say "these two sets are not of the same cardinality", but also "this set is of higher cardinality than that one".